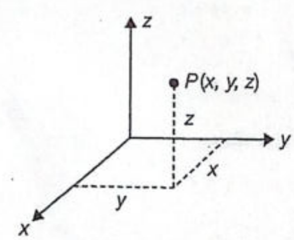
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ (Faraday's law)

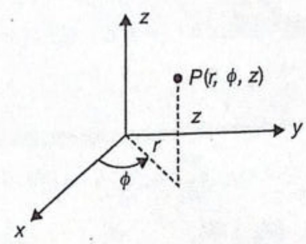
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$ (Ampere's law supplemented by Maxwell's displacement current)

$\nabla \cdot \mathbf{D} = \rho$ (Gauss's law for the electric field)

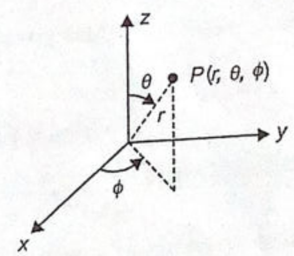
$\nabla \cdot \mathbf{B} = 0$ (Gauss's law for the magnetic field)



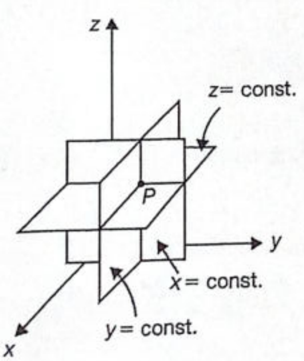
(*a*) Cartesian



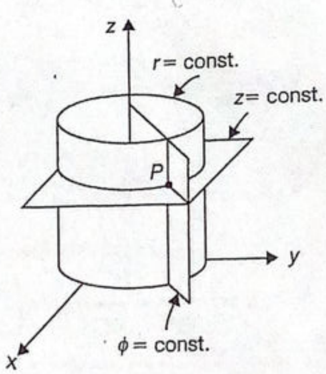
(*b*) Cylindrical



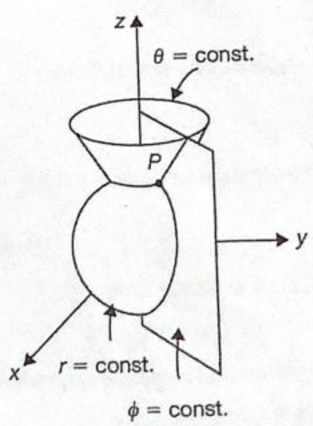
(*c*) Spherical



(*a*) Cartesian



(*b*) Cylindrical



(*c*) Spherical

2.4 Vector Algebra

1. Vectors may be added and subtracted.

$$\mathbf{A} \pm \mathbf{B} = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) \pm (B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z)$$
$$= (A_x \pm B_x)\mathbf{a}_x + (A_y \pm B_y)\mathbf{a}_y + (A_z \pm B_z)\mathbf{a}_z$$

2. The associative, distributive, and commutative laws apply.

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$
$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B} \qquad (k_1 + k_2)\mathbf{A} = k_1\mathbf{A} + k_2\mathbf{A}$$
$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

3. The *dot product* of two vectors is, by definition,

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta \qquad \text{(read "A dot B")}$$

where θ is the smaller angle between **A** and **B**. In Example 3 it is shown that

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

which gives, in particular, $|A| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$.

TABLE 1-5 Some Useful Vector Operators and Identities

(1) Cartesian vector: $\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$

(2) Time-derivative of a vector: $\dfrac{\partial \mathbf{A}}{\partial t} = \dfrac{\partial A_x}{\partial t}\mathbf{a}_x + \dfrac{\partial A_y}{\partial t}\mathbf{a}_y + \dfrac{\partial A_z}{\partial t}\mathbf{a}_z$

(3) Dot product of two vectors: $\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$

(4) Cross product of two vectors: $\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\,\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z$

(5) Del operator: $\nabla \equiv \dfrac{\partial(\,)}{\partial x}\mathbf{a}_x + \dfrac{\partial(\,)}{\partial y}\mathbf{a}_y + \dfrac{\partial(\,)}{\partial z}\mathbf{a}_z$

(6) Gradient of a scalar field: $\nabla F = \dfrac{\partial F}{\partial x}\mathbf{a}_x + \dfrac{\partial F}{\partial y}\mathbf{a}_y + \dfrac{\partial F}{\partial z}\mathbf{a}_z$

(7) Divergence of a vector field: $\nabla \cdot \mathbf{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$

(8) Curl of a vector field: $\nabla \times \mathbf{A} = \left(\dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}\right)\mathbf{a}_z$

(9) Laplacian (divergence of gradient) of a scalar field: $\nabla^2 F \equiv \nabla \cdot \nabla F = \dfrac{\partial^2 F}{\partial x^2} + \dfrac{\partial^2 F}{\partial y^2} + \dfrac{\partial^2 F}{\partial z^2}$

(10) Curl curl of a vector field: $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

(11) Vector identities:
  (a) Divergence of the curl is zero $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
  (b) Curl of the gradient is zero $\nabla \times (\nabla F) = 0$

TABLE 5-1 Summary of Vector Operations

COORDINATE SYSTEM	OPERATOR	MATHEMATICAL FORMULA
Cartesian	Del operator	$\nabla = \frac{\partial(\ )}{\partial x}\mathbf{a}_x + \frac{\partial(\ )}{\partial y}\mathbf{a}_y + \frac{\partial(\ )}{\partial z}\mathbf{a}_z$
	Gradient	$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$
	Divergence	$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
	Curl	$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z$
	Laplacian	$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$
Cylindrical	Gradient	$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{\partial V}{r\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z$
	Divergence	$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$
	Curl	$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{a}_\phi + \frac{1}{r}\left[\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right]\mathbf{a}_z$
	Laplacian	$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$
Spherical	Gradient	$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$
	Divergence	$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$
	Curl	$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\mathbf{a}_\phi$
	Laplacian	$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$

TABLE 1-6 Derivation of the Plane Wave Equation

Faraday's law	Ampere's law
$\Downarrow$	$\Downarrow$
$\nabla \times \mathbf{E} = -\mu \dfrac{\partial \mathbf{H}}{\partial t}$	$\nabla \times \mathbf{H} = \epsilon \dfrac{\partial \mathbf{E}}{\partial t}$
$\Downarrow$	$\Downarrow$
for plane waves $E_y = E_z = 0$ $H_x = H_z = 0$	for plane waves $E_y = E_z = 0$ $H_x = H_z = 0$
$\Downarrow$	$\Downarrow$
$\dfrac{\partial E_x}{\partial z} = -\mu \dfrac{\partial H_y}{\partial t}$	$\dfrac{\partial H_y}{\partial z} = -\epsilon \dfrac{\partial E_x}{\partial t}$
$\Downarrow$	$\Downarrow$
differentiation with respect to z	differentiation with respect to t
$\Downarrow$	$\Downarrow$
$\dfrac{\partial^2 E_x}{\partial z^2} = -\mu \dfrac{\partial^2 H_y}{\partial t \partial z}$	$\dfrac{\partial^2 H_y}{\partial z \partial t} = -\epsilon \dfrac{\partial^2 E_x}{\partial t^2}$
$\searrow$	$\swarrow$

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2}}$$

TABLE 1-7 Classical Wave Equations in Time and Phasor Domains

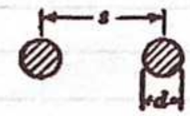
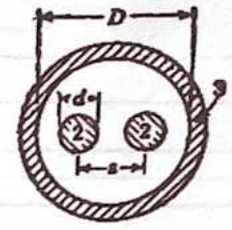
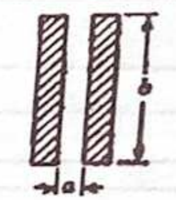
Time domain:	$\nabla^2 \mathbf{E} = \mu\epsilon \dfrac{\partial^2 \mathbf{E}}{\partial t^2}$	$\nabla^2 \mathbf{H} = \mu\epsilon \dfrac{\partial^2 \mathbf{H}}{\partial t^2}$
Phasor domain:	$\nabla^2 \mathbf{E} + \omega^2 \mu\epsilon \mathbf{E} = 0$	$\nabla^2 \mathbf{H} + \omega^2 \mu\epsilon \mathbf{H} = 0$
where ∇^2 is the Laplacian operator	$\nabla^2 \mathbf{E} = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \right) \mathbf{E}$	

These are waves which travel at a speed of $u = 1/\sqrt{\mu\epsilon}$, which is the speed of light in the given medium. To derive the wave equations, start with Maxwell's equations in a medium with permeability μ and permittivity ϵ, containing no charges or currents ($\rho = 0$ and $J = 0$). Then proceed as shown in Table 1-8 for the case of the **E** field.

TABLE 1-8 Derivation of the Wave Equation for the Electric Field in Source-Free Media

Step 1. Take the curl of both sides in Faraday's law	$\nabla \times (\nabla \times \mathbf{E}) = -\mu \dfrac{\partial}{\partial t} (\nabla \times \mathbf{H})$
Step 2. Substitute for $\nabla \times \mathbf{H}$ from Ampere's law	$\nabla \times (\nabla \times \mathbf{E}) = -\mu\epsilon \dfrac{\partial^2 \mathbf{E}}{\partial t^2}$
Step 3. Note that the curl of **E** is	$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$
Step 4. Also note that the divergence of **E** is zero.	$\nabla \cdot \mathbf{E} = 0$
Step 5. Therefore,	$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}$
Step 6. Substitute the result of Step 5 into Step 2 to find	$\boxed{\nabla^2 \mathbf{E} = \mu\epsilon \dfrac{\partial^2 \mathbf{E}}{\partial t^2}}$

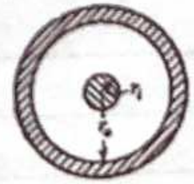
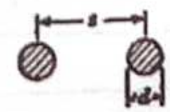
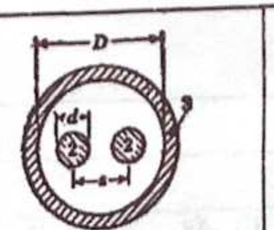
Transmission Line Parameters

			$p = \frac{s}{d}$ $q = \frac{s}{D}$ 	 Formulas for $a \ll b$
Capacitance C, farads/meter	$\frac{2\pi\epsilon}{\ln\left(\frac{r_0}{r_i}\right)}$	$\frac{\pi\epsilon}{\cosh^{-1}\left(\frac{s}{d}\right)}$	-----	$\frac{\epsilon b}{a}$
External inductance L, henrys/meter	$\frac{\mu}{2\pi}\ln\left(\frac{r_0}{r_i}\right)$	$\frac{\mu}{\pi}\cosh^{-1}\left(\frac{s}{d}\right)$	-----	$\mu\frac{a}{b}$
Conductance G, siemens/meter	$\frac{2\pi\sigma}{\ln\left(\frac{r_0}{r_i}\right)} = \frac{2\pi\omega\epsilon''}{\ln\left(\frac{r_0}{r_i}\right)}$	$\frac{\pi\sigma}{\cosh^{-1}\left(\frac{s}{d}\right)} = \frac{\pi\omega\epsilon''}{\cosh^{-1}\left(\frac{s}{d}\right)}$	-----	$\frac{\sigma b}{a} = \frac{\omega\epsilon'' b}{a}$
Resistance R, ohms/meter	$\frac{R_s}{2\pi}\left(\frac{1}{r_0} + \frac{1}{r_i}\right)$	$R_{DC} = \frac{2}{\sigma S}$ $\frac{2R_s}{\pi d}\left[\frac{s/d}{\sqrt{(s/d)^2 - 1}}\right]$	$\frac{2R_{s2}}{\pi d}\left[1 + \frac{1+2p^2}{4p^4}(1-4q^2)\right] + \frac{8R_{s3}}{\pi D}q^2\left[1 + q^2 - \frac{1+4p^2}{8p^4}\right]$	$\frac{2R_s}{b}$
Internal inductance L_i, henrys/meter (for high frequency)	←	$\frac{R}{\omega}$		→

$$R_s \equiv \frac{1}{\sigma\delta} \qquad \delta \equiv \frac{1}{\sqrt{\pi f \mu \sigma}} \quad \text{(skin depth)}$$

conductivity

Transmission Line Parameters

			$p = \frac{s}{d}$ $q = \frac{s}{D}$ 	 Formulas for $a \ll b$
Characteristic impedance at high frequency Z_0, ohms	$\frac{\eta}{2\pi} \ln\left(\frac{r_0}{r_i}\right)$	$\frac{\eta}{\pi} \cosh^{-1}\left(\frac{s}{d}\right)$	$\frac{\eta}{\pi}\left\{\ln\left[2p\left(\frac{1-q^2}{1+q^2}\right)\right] - \frac{1+4p^2}{16p^4}(1-4q^2)\right\}$	$\eta \frac{a}{b}$
Z_0 for air dielectric	$60 \ln\left(\frac{r_0}{r_i}\right)$	$120 \cosh^{-1}\left(\frac{s}{d}\right) \cong 120 \ln\left(\frac{2s}{d}\right)$ if $s/d \gg 1$	$120\left\{\ln\left[2p\frac{(1-q^2)}{(1+q^2)}\right] - \frac{1+4p^2}{16p^4}(1-4q^2)\right\}$	$120\pi \frac{a}{b}$
Attenuation due to conductor α_c	$\frac{R}{2Z_0}$			
Attenuation due to dielectric α_d	$\frac{GZ_0}{2} = \frac{\sigma\eta}{2}$			
Total attenuation dB/meter	$8.686(\alpha_c + \alpha_d)$			
Phase constant for low-loss lines β	$\omega\sqrt{\mu\epsilon'} = \frac{2\pi}{\lambda}$			

$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \alpha + j\beta$

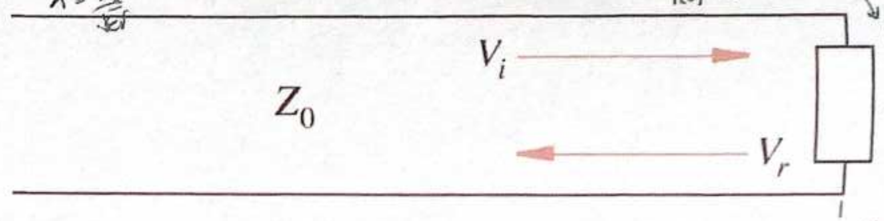
$1\,NB/m = 8.686\,dB/m$ — PL rate

$v = \frac{\omega}{\beta} = f\lambda$, $\beta = \frac{2\pi}{\lambda}$, $f = \frac{c}{\lambda}$ (3×10^8)

STANDING WAVES ON TRANSMISSION LINES

$$V_i(z) = V_{i0}e^{-j\beta z}$$

$$V_r(z) = V_{r0}e^{j\beta z}$$



$\frac{P_t}{P_i} = 1-|\Gamma|^2$, $\frac{P_r}{P_i} = |\Gamma|^2$

$P_r = P_i|\Gamma|^2$, $P_t = P_i(1-|\Gamma|^2)$

$PL = 10\log_{10}\left(\frac{1}{1-|\Gamma|^2}\right)$

if loss in line.

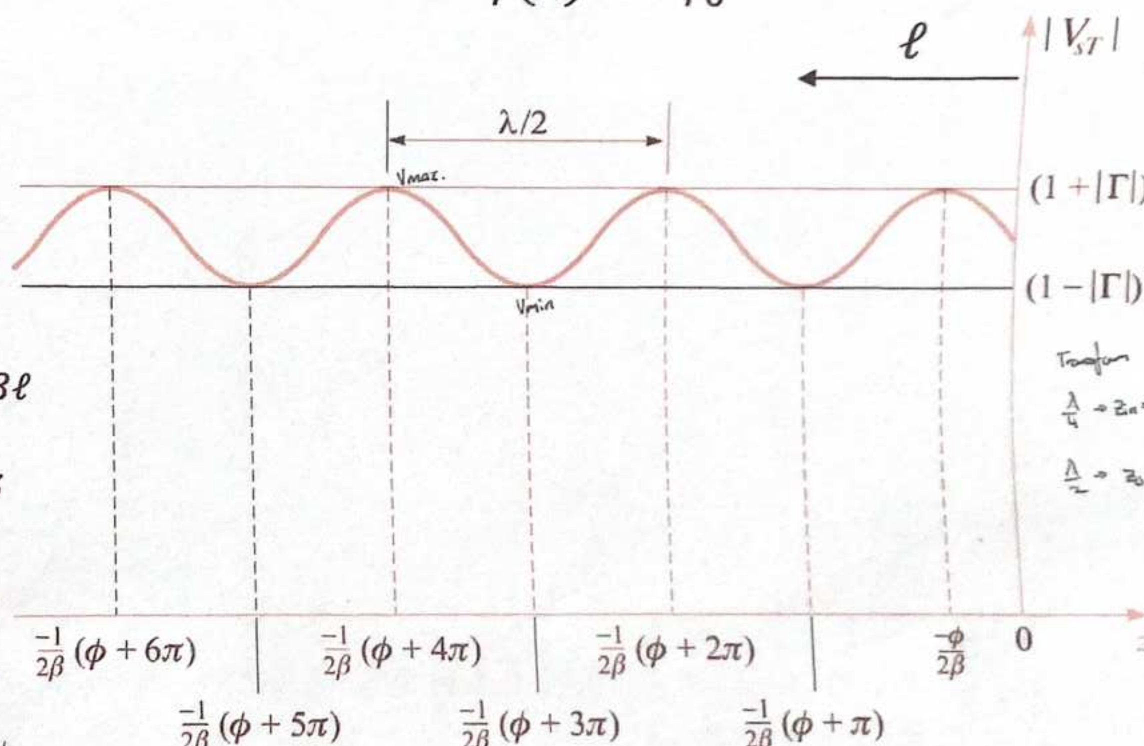
$$S = \frac{V_{\max}}{V_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

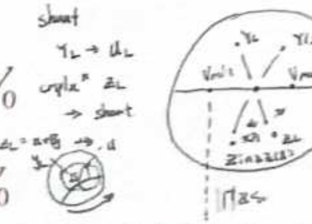
$|\Gamma| = \frac{S-1}{S+1}$, consider if load is at max or min

$$\Gamma_0 = \frac{V_r}{V_i} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\beta = \frac{2\pi}{\lambda}$$

$Z_0 = \sqrt{\frac{(R+j\omega L)}{(G+j\omega L)}}$





Transform

$\frac{\lambda}{4} \to Z_{in} = \frac{Z_0^2}{Z_L}$

$\frac{\lambda}{2} \to Z_0 = Z_L$

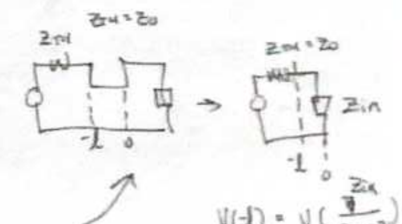
$$\Gamma(\ell) = \Gamma_0 e^{-j2\beta\ell}$$

$$\Gamma(z) = \Gamma_0 e^{j2\beta z}$$

$PL = 10\log_{10}\left(\frac{P(0)}{P(z)}\right) = 8.69\,\alpha z$

$\frac{P(z)}{P(0)} = 10^{-\frac{PL}{10}}$, $PL = PL_{rate}\cdot\ell$

$\frac{P(z)}{P(0)}\times100 \Rightarrow$ % of input reach output.



at z

$$Z_{in}(z) = Z_0\frac{Z_L - jZ_0\tan(\beta z)}{Z_0 - jZ_L\tan(\beta z)}$$

from z

$$Z_{in}(\ell) = Z_0\frac{Z_L + jZ_0\tan(\beta\ell)}{Z_0 + jZ_L\tan(\beta\ell)}$$

$P_{Zin}^{avg} = \frac{1}{2}|I|^2 R$

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PHYSICAL CONSTANTS

Permittivity of free space: $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

Electron charge: $e = 1.602 \times 10^{-19}$ C

Speed of light in vacuum: $c = 2.998 \times 10^{8}$ m/s

Permeability of free space: $\mu_0 = 4\pi \times 10^{-7}$ H/m

Electron mass: $m = 9.109 \times 10^{-31}$ kg

ELECTROSTATIC PRINCIPLES

Coulomb's Law: $\mathbf{F}_2 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1 q_2}{R_{12}^2}\mathbf{a}_{12}$ $\qquad$ $\mathbf{a}_{12} = \dfrac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \dfrac{\mathbf{R}_{12}}{R_{12}} = \dfrac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$

Point Charge Q at O: $\mathbf{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\mathbf{a}_r,\ V = \dfrac{Q}{4\pi\varepsilon_0 r}$ $\qquad$ (r comes from spherical coords)

Line Charge, density ρ_L, on z-axis: $\mathbf{E} = \dfrac{\rho_L}{2\pi\varepsilon_0}\left(\dfrac{\mathbf{a}_\rho}{\rho}\right),\ V = \dfrac{\rho_L}{2\pi\varepsilon_0}\ln\left(\dfrac{1}{\rho}\right)$ $\qquad$ (ρ comes from cylindrical coords)

Sheet Charge, density ρ_S, on $z = 0$: $\mathbf{E} = \pm\dfrac{\rho_S}{2\varepsilon_0}\mathbf{a}_z,\ V = -\dfrac{\rho_S|z|}{2\varepsilon_0}$ $\qquad$ (Both ρ_S and ρ_L must be constant here.)

Electric Flux Density: $\mathbf{D} = \varepsilon\mathbf{E}$ $\qquad$ ($\varepsilon = \varepsilon_0\varepsilon_r$ in general; $\varepsilon_r = 1$ in free space)

Gauss's Law, I: $Q_{\text{enc}} = \Psi$, where $\qquad$ $\Psi = \oiint_S \mathbf{D} \bullet \widehat{\mathbf{n}}\, dS$ is net outward flux

Gauss's Law, II: $Q_{\text{enc}} = \iiint_{\mathcal{V}} \rho_v\, dv$, where $\qquad$ $\rho_v = \nabla \bullet \mathbf{D}$ gives charge density

Electric field and potential: $\mathbf{E} = -\nabla V$ $\qquad$ $V(B) - V(A) = -\int_A^B \mathbf{E} \bullet d\mathbf{L}$ (path indep)

Generalized Poisson Equation: $\nabla \bullet (\varepsilon \nabla V) = -\rho_v$ $\qquad$ (Case $\rho_v = 0$, $\varepsilon =$ const is Laplace's Equation.)

Energy in Electrostatic Field: $W_E = \frac{1}{2}\iiint_{\mathcal{R}} \mathbf{D}\bullet\mathbf{E}\, dv = \frac{1}{2}\iiint_{\mathcal{R}} \varepsilon\,|\mathbf{E}|^2\, dv$

CONDUCTORS, CURRENT, RESISTANCE

Ideal conductor ("$\sigma \to \infty$"): $\mathbf{E} = \mathbf{0}$ $\qquad$ $V =$ const.

Ideal conductor boundary: $\mathbf{E} \parallel \widehat{\mathbf{n}}$ $\qquad$ $\rho_S = \mathbf{D} \bullet \widehat{\mathbf{n}}$

Current and conductivity: $\mathbf{J} = \sigma\mathbf{E}$ "Ohm's Law I" $\qquad$ $I = \iint_S \mathbf{J} \bullet \widehat{\mathbf{n}}\, dS$

$\mathbf{J} = \rho_v \mathbf{v}$ $\qquad$ $\nabla \bullet \mathbf{J} = -\dfrac{\partial \rho_v}{\partial t}$

Simple Resistor (length L, constant cross-section S, constant conductivity σ): $R = \dfrac{L}{\sigma S}$

Fancy Resistor (all current from A to B crosses surface S—"Ohm's Law II"): $R = \dfrac{|\Delta V|}{|I|} = \dfrac{\left|-\int_A^B \mathbf{E} \bullet d\mathbf{L}\right|}{\left|\iint_S \mathbf{J} \bullet \widehat{\mathbf{n}}\, dS\right|}$

CAPACITORS AND DIELECTRICS

Permittivity: $\varepsilon = \varepsilon_r \varepsilon_0$ $\qquad$ (Gauss's Law still works, as above)

Polarization: $\mathbf{P} = \mathbf{D} - \varepsilon_0\mathbf{E}$

Simple Capacitor (parallel plates of area S, separation d): $C = \dfrac{\varepsilon S}{d}$ stores $W_E = \frac{1}{2}CV^2$ Joules

Fancy Capacitor (surface S is one plate; points A, B on opposite plates): $C = \dfrac{|Q|}{|\Delta V|} = \dfrac{\left|\iint_S \mathbf{D} \bullet \widehat{\mathbf{n}}\, dS\right|}{\left|-\int_A^B \mathbf{E} \bullet d\mathbf{L}\right|}$

Dielectric interface with normal $\mathbf{n}$: $\mathbf{D}_1 \bullet \mathbf{n} = \mathbf{D}_2 \bullet \mathbf{n}$ AND $\mathbf{E}_1 \times \mathbf{n} = \mathbf{E}_2 \times \mathbf{n}$

MAGNETOSTATICS

Biot-Savart Law: $d\mathbf{H} = \dfrac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$ $\quad \mathbf{H} = \displaystyle\int \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$

Current I flowing in filament $\rho = 0$, direction $\mathbf{a}_z$: $\mathbf{H} = \dfrac{I}{2\pi\rho}\mathbf{a}_\phi$; or, for segment, $\mathbf{H} = \dfrac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\,\mathbf{a}_\phi$

Current sheet with density $\mathbf{K}$ [A/m], normal $\widehat{\mathbf{n}}$: $\mathbf{H} = \frac{1}{2}\mathbf{K} \times \widehat{\mathbf{n}}$ $\quad I = \displaystyle\int \mathbf{K} \bullet dw$

Current crossing surface S, from current density $\mathbf{J}$: $I = \displaystyle\iint_S \mathbf{J} \bullet d\mathbf{S}$ $\quad \mathbf{J} = \nabla \times \mathbf{H}$

Ampère's Circuital Law (ACL): $I = \displaystyle\oint \mathbf{H} \bullet d\mathbf{L}$ $\quad$ (compare Stokes's Theorem)

Magnetic Flux Density: $\mathbf{B} = \mu\mathbf{H}$ $\quad \mu = \mu_r\mu_0$

Magnetic Flux (Wb): $\Phi = \displaystyle\iint_S \mathbf{B} \bullet d\mathbf{S}$ $\quad \displaystyle\oiint_S \mathbf{B} \bullet d\mathbf{S} = 0$

Energy in Steady Magnetic Field: $W_H = \frac{1}{2}\displaystyle\iiint_{\mathcal{R}} \mathbf{B}\bullet\mathbf{H}\,dv = \frac{1}{2}\iiint_{\mathcal{R}} \mu\,|\mathbf{H}|^2\,dv$

Magnetic Force on Moving Charge: $\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$ $\quad \mathbf{J} = \mathbf{v}\rho_v$

Magnetic Force on Current Filament: $d\mathbf{F} = I\,d\mathbf{L} \times \mathbf{B}$ $\quad \mathbf{F} = \displaystyle\int_C I\,d\mathbf{L}\times\mathbf{B} = -\int_C I\mathbf{B}\times d\mathbf{L}$

Magnetic Force on Current Sheet or Cloud: $d\mathbf{F} = (\mathbf{K}\,dS) \times \mathbf{B}$ $\quad d\mathbf{F} = (\mathbf{J}\,dv) \times \mathbf{B}$

Magnetic Dipole Moment ($\mathbf{m} = \mathbf{p}_m$): $d\mathbf{m} = I\,d\mathbf{S}$ $\quad \mathbf{m} = NIS\widehat{\mathbf{n}}$

Magnetic Torque on Given Dipole: $\vec{\tau} = \mathbf{m} \times \mathbf{B}$ $\quad |\vec{\tau}| = NI\,|\mathbf{B}|\,|\mathbf{S}|$, if $\mathbf{B} \perp \mathbf{S}$

Review: Force $\mathbf{F}$ with moment arm $\mathbf{R}$ gives torque: $\vec{\tau} = \mathbf{R} \times \mathbf{F}$

INDUCTORS AND MAGNETIC MATERIALS

Permeability: $\mu = \mu_r\mu_0$

Simple inductor (N filaments, current I in each): $L = \dfrac{N\Phi}{I}$ $\quad$ stores $W_H = \frac{1}{2}LI^2$ Joules

Mutual Inductance: $M_{12} = \dfrac{N_2\Phi_{12}}{I_1} = \dfrac{N_1\Phi_{21}}{I_2} = M_{21}$

Material interface with normal $\mathbf{n}$: $\mathbf{B}_1 \bullet \mathbf{n} = \mathbf{B}_2 \bullet \mathbf{n}$ $\quad \mathbf{H}_1 \times \mathbf{n} = \mathbf{H}_2 \times \mathbf{n}$

MAGNETIC CIRCUITS

Magnetomotive force (simple setup—N turns, current I): $V_m = NI$

Magnetomotive force (general—filament from A to B): $V_m(B) - V_m(A) = -\displaystyle\int_A^B \mathbf{H} \bullet d\mathbf{L}$ $\quad$ (path restrictions apply)

Reluctance (cross-section S, length ℓ): $\mathcal{R} = \dfrac{V_m}{\Phi} = \dfrac{\ell}{\mu S}$ $\quad$ (integral defining Φ shown above)

Air-gap force (cross-section S): $\mathbf{F} = \dfrac{1}{2\mu_0}|\mathbf{B}|^2 S\,\widehat{\mathbf{n}}$

MAXWELL'S EQUATIONS (POINT FORM, GENERAL CASE—set $\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}$ and $\frac{\partial \mathbf{D}}{\partial t} = \mathbf{0}$ in static situations)

$\nabla \bullet \mathbf{D} = \rho_v$ $\quad \nabla \bullet \mathbf{B} = 0$ $\quad \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ $\quad \nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$

TIME-VARYING FIELDS

Faraday's Law (case of $N = 1$ current filament): $\text{emf} = -\dfrac{d\Phi}{dt} = -\displaystyle\iint_S \frac{\partial \mathbf{B}}{\partial t} \bullet \widehat{\mathbf{n}}\,dS$ $\quad$ (units: Volts)

$\text{emf} = \displaystyle\oint_C \mathbf{E} \bullet d\mathbf{L}$ $\quad$ (loop shape matters!)

VECTOR IDENTITIES

For $\mathbf{u} = u_x\mathbf{a}_x + u_y\mathbf{a}_y + u_z\mathbf{a}_z$, $\mathbf{v} = v_x\mathbf{a}_x + v_y\mathbf{a}_y + v_z\mathbf{a}_z$, $\mathbf{w} = w_x\mathbf{a}_x + w_y\mathbf{a}_y + w_z\mathbf{a}_z$,

$\mathbf{u} \bullet \mathbf{v} = u_xv_x + u_yv_y + u_zv_z = |\mathbf{u}|\,|\mathbf{v}|\cos(\theta), \quad 0 \le \theta \le \pi$

$|\mathbf{u}| = \sqrt{\mathbf{u} \bullet \mathbf{u}} = \sqrt{u_x^2 + u_y^2 + u_z^2}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \langle u_yv_z - u_zv_y,\, u_zv_x - u_xv_z,\, u_xv_y - u_yv_x \rangle$$

$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta$

$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \bullet (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})$

$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\mathbf{w}$

DISTANCES AND PROJECTIONS

From point (x_0, y_0, z_0) to plane $Ax + By + Cz = D$: $\quad s = \dfrac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$

$\mathbf{F} = \mathbf{proj_u}(\mathbf{F}) + \mathbf{orth_u}(\mathbf{F})$

$\mathbf{proj_u}(\mathbf{F}) = \left(\dfrac{\mathbf{F} \bullet \mathbf{u}}{\mathbf{u} \bullet \mathbf{u}}\right)\mathbf{u}$

DERIVATIVE IDENTITIES – valid for smooth scalar-valued ϕ, ψ and smooth vector-valued $\mathbf{F}$, $\mathbf{G}$

$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \bullet (\phi\mathbf{F}) = (\nabla\phi) \bullet \mathbf{F} + \phi(\nabla \bullet \mathbf{F})$

$\nabla \times (\phi\mathbf{F}) = (\nabla\phi) \times \mathbf{F} + \phi(\nabla \times \mathbf{F})$

$\nabla \times (\nabla\phi) = \mathbf{0} \qquad (\mathbf{curl}\,\mathbf{grad} = \mathbf{0})$

$\nabla \bullet (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \bullet \mathbf{G} - \mathbf{F} \bullet (\nabla \times \mathbf{G})$

$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \bullet \mathbf{G}) - \mathbf{G}(\nabla \bullet \mathbf{F}) - (\mathbf{F} \bullet \nabla)\mathbf{G} + (\mathbf{G} \bullet \nabla)\mathbf{F}$

$\nabla(\mathbf{F} \bullet \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \bullet \nabla)\mathbf{G} + (\mathbf{G} \bullet \nabla)\mathbf{F}$

$\nabla \bullet (\nabla \times \mathbf{F}) = 0 \qquad (\text{div}\,\mathbf{curl} = 0)$

$$\nabla^2\phi(x,y,z) = \nabla \bullet \nabla\phi(x,y,z) = \text{div}\,\mathbf{grad}\,\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

SURFACE NORMALS AND AREA ELEMENTS

For any oriented surface normal $\mathbf{n} \neq \mathbf{0}$, $\quad d\mathbf{S} = \widehat{\mathbf{n}}\,dS = \dfrac{\mathbf{n}}{|\mathbf{n} \bullet \mathbf{a}_z|}\,dx\,dy = \dfrac{\mathbf{n}}{|\mathbf{n} \bullet \mathbf{a}_y|}\,dx\,dz = \dfrac{\mathbf{n}}{|\mathbf{n} \bullet \mathbf{a}_x|}\,dy\,dz, \qquad dS = |d\mathbf{S}|$

Graph Surface $z = f(x,y)$: $\quad$ normal $\mathbf{n} = \pm\left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, -1 \right\rangle \quad \widehat{\mathbf{n}}\,dS = \pm\left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, -1 \right\rangle dx\,dy$

Level Surface $G(x,y,z) = 0$: $\quad$ normal $\mathbf{n} = \pm\nabla G(x,y,z) \quad$ (choose sign to orient)

Parametric Surface $\langle x,y,z \rangle = \mathbf{R}(u,v)$: $\quad d\mathbf{S} = \pm\left(\dfrac{\partial \mathbf{R}}{\partial u} \times \dfrac{\partial \mathbf{R}}{\partial v}\right) du\,dv \quad$ (choose sign to orient; $\widehat{\mathbf{n}} = \dfrac{d\mathbf{S}}{|d\mathbf{S}|}$)

CARTESIAN COORDINATES (x, y, z)

Line Element: $d\mathbf{L} = \mathbf{a}_x\,dx + \mathbf{a}_y\,dy + \mathbf{a}_z\,dz$

Volume Element: $dv = dx\,dy\,dz$

Scalar field: $f(x,y,z)$

Vector field: $\mathbf{F}(x,y,z) = F_x\,\mathbf{a}_x + F_y\,\mathbf{a}_y + F_z\,\mathbf{a}_z$

Differential operator ∇: $\quad \nabla = \mathbf{a}_x\dfrac{\partial}{\partial x} + \mathbf{a}_y\dfrac{\partial}{\partial y} + \mathbf{a}_z\dfrac{\partial}{\partial z} = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle$

Gradient: $\nabla f = \dfrac{\partial f}{\partial x}\mathbf{a}_x + \dfrac{\partial f}{\partial y}\mathbf{a}_y + \dfrac{\partial f}{\partial z}\mathbf{a}_z$

Divergence: $\nabla \bullet \mathbf{F}(x,y,z) = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}$

$$\text{Curl: } \nabla \times \mathbf{F} = \mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Laplacian: $\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

POLAR AND CYLINDRICAL COORDINATES (ρ, ϕ, z)

Transformation: $x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$

Local basis: $\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y, \quad \mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y, \quad \mathbf{a}_z = \mathbf{a}_z$

Surface element (on $\rho = a$): $d\mathbf{S} = \pm a\,\mathbf{a}_\rho\,d\phi\,dz$

Surface element (on $z =$const.): $d\mathbf{S} = \pm\rho\,\mathbf{a}_z\,d\rho\,d\phi$

Line Element: $d\mathbf{L} = \mathbf{a}_\rho\,d\rho + \rho\mathbf{a}_\phi\,d\phi + \mathbf{a}_z\,dz$

Volume element: $dv = \rho\,d\rho\,d\phi\,dz$

Scalar field: $f(\rho, \phi, z)$

Vector field: $\mathbf{F}(\rho, \phi, z) = F_\rho\,\mathbf{a}_\rho + F_\phi\,\mathbf{a}_\phi + F_z\,\mathbf{a}_z$

$$\nabla f = \frac{\partial f}{\partial\rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\mathbf{a}_\phi + \frac{\partial f}{\partial z}\mathbf{a}_z$$

$$\nabla\bullet\mathbf{F} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}$$

$$\nabla\times\mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{a}_\rho & \rho\mathbf{a}_\phi & \mathbf{a}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$$

SPHERICAL COORDINATES (r, θ, ϕ)

Transformation: $x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$

Local basis: $\mathbf{a}_r = \sin\theta\cos\phi\,\mathbf{a}_x + \sin\theta\sin\phi\,\mathbf{a}_y + \cos\theta\,\mathbf{a}_z$, $\quad \mathbf{a}_\theta = \cos\theta\cos\phi\,\mathbf{a}_x + \cos\theta\sin\phi\,\mathbf{a}_y - \sin\theta\,\mathbf{a}_z$,

$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$

Volume element: $dv = r^2\sin\theta\,dr\,d\theta\,d\phi$

Surface area element (on $r = a$): $d\mathbf{S} = \pm a^2\sin\theta\,\mathbf{a}_r\,d\theta\,d\phi$

Line Element: $d\mathbf{L} = \mathbf{a}_r\,dr + r\mathbf{a}_\theta\,d\theta + r\sin\theta\mathbf{a}_\phi\,d\phi$

Scalar field: $f(r, \theta, \phi)$

Vector field: $\mathbf{F}(r, \theta, \phi) = F_r\,\mathbf{a}_r + F_\theta\,\mathbf{a}_\theta + F_\phi\,\mathbf{a}_\phi$

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{a}_\phi$$

$$\nabla\bullet\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

INTEGRATING DERIVATIVES: THE FUNDAMENTAL THEOREM OF CALCULUS (FTC)

Line-integral form: $$\int_C \nabla g\bullet d\mathbf{L} = \int_C \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz = g_{\text{final}} - g_{\text{initial}}$$

Stokes's Theorem: $$\iint_S (\nabla\times\mathbf{G})\bullet d\mathbf{S} = \oint_C \mathbf{G}\bullet d\mathbf{L} = \oint_C G_x\,dx + G_y\,dy + G_z\,dz$$

Divergence Theorem: $$\iiint_{\mathcal{R}} \nabla\bullet\mathbf{G}\,dv = \oiint_S \mathbf{G}\bullet\hat{\mathbf{n}}\,dS$$

DEFINITE INTEGRALS

$$\int_0^{\pi/2}\sin x\,dx = \int_0^{\pi/2}\cos x\,dx = 1 \qquad \int_0^{\pi/2}\sin^3 x\,dx = \int_0^{\pi/2}\cos^3 x\,dx = \frac{2}{3} \qquad \int_0^{\pi/2}\sin^5 x\,dx = \int_0^{\pi/2}\cos^5 x\,dx = \frac{8}{15}$$

$$\int_0^{\pi/2}\sin^2 x\,dx = \int_0^{\pi/2}\cos^2 x\,dx = \frac{\pi}{4} \qquad \int_0^{\pi/2}\sin^4 x\,dx = \int_0^{\pi/2}\cos^4 x\,dx = \frac{3\pi}{16} \qquad \int_0^{\pi/2}\sin^6 x\,dx = \int_0^{\pi/2}\cos^6 x\,dx = \frac{5\pi}{32}$$

INDEFINITE INTEGRALS

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) \qquad \int\tan x\,dx = \ln|\sec x| \qquad \int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \quad (a>0)$$

$$\int\sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x \qquad \int\cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x \qquad \int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) \quad (a>0)$$

$$\int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) \quad (a>0) \qquad \int\frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}} \qquad \int\frac{dx}{(x^2\pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2\pm a^2}}$$

$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \quad (a>0) \qquad \int\frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right|$$

Adapted from R. A. Adams, *Calculus, A Complete Course*, Addison-Wesley, 2003.

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}} = \frac{v_1}{v_2}\,.$$

$$E_\theta = E_0\,\frac{e^{-jkr}}{r}$$

$$\frac{E_0^r}{E_0^i} = \Gamma_\perp = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$\frac{E_0^t}{E_0^i} = \tau_\perp = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$\frac{E_0^r}{E_0^i} = \Gamma_\parallel = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\frac{E_0^t}{E_0^i} = \tau_\parallel = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\theta_B = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

$$S_{\text{avg}} = \frac{1}{2}|E_\theta|^2/\eta_0 \quad = \tfrac{1}{2}\,|E_x| \cdot \frac{1}{\eta}$$

$$\Gamma_{TE}(E) = \Gamma_{TE}(H) = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$\Gamma_{TM}(E) = \Gamma_{TM}(H) = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$R_\Omega = \frac{\rho L}{A} = \frac{L}{\sigma A}$$

$$\mathbf{E} = \frac{1}{j\omega\epsilon_0}\nabla\times\mathbf{H}$$

$$\nabla\times\mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\beta = \omega\sqrt{\mu\epsilon}$$

skin depth $$\delta = \frac{1}{\sqrt{\pi f\mu\sigma}}$$

$$E_x = E_0\,e^{j(\omega t - \beta z)}$$

$$\sin t = \cos(t - \pi/2)$$

$$\sin^2 t + \cos^2 t = 1$$

$$\sin 2t = 2\sin t\cos t$$

$$\cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 = 1 - 2\sin^2 t$$

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$2\cos^2(a/2) = 1 - \cos a$$

$$2\sin^2(a/2) = 1 + \cos a$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi\Omega$$

lossless.

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

impedance.

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^\circ$$

$$\eta_w(z) = \eta_1\,\frac{\eta_2 - j\eta_1\tan\beta_1 z}{\eta_1 - j\eta_2\tan\beta_1 z}$$

$$\tan\delta = \frac{\sigma}{\omega\epsilon}$$ loss tangent.

$$v_g = v\sin\theta_m$$

$$v_g v_p = v^2$$

$$k^2 = k_c^2 + k_z^2$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

ELEC 311 - Electromagnetic Fields & Waves
Formula Sheet

$\text{emf} = -N\frac{d\Phi}{dt} = \oint E \cdot dL = -\frac{d}{dt}\int B \cdot d\vec{S}$

$\Phi = \int_s \vec{B} \cdot d\vec{S}$

$F = Q\vec{u} \times \vec{B}$

$E_m = \vec{v} \times \vec{B}$

$v = \frac{dy}{dt}$

$$
\begin{aligned}
\mathbf{D} &= \epsilon_r\epsilon_0\mathbf{E} \\
\mathbf{B} &= \mu_r\mu_0\mathbf{H} \\
\mathbf{J} &= \sigma\mathbf{E}
\end{aligned}
$$

$V = -\beta\vec{v}d$ (β is const.)

c of int = 0 if no DC

$$\nabla \times \mathbf{A} = \mu\mathbf{H}$$

$\vec{K} = \vec{H}_{T_1} \times \hat{a}_n$

$$V = \int_v \frac{\rho\, dv}{4\pi\epsilon_0 R}$$

$V = \frac{kq}{d}$

$$\mathbf{A} = \oint \frac{\mu I d\ell}{4\pi R}$$

t of change $(t') = t - \frac{d}{c}$

$\vec{B}^2 = \vec{B}_T^2 + \vec{B}_N^2$

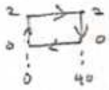
$\vec{B}_{N_1} = \vec{B}_{N_2} = \vec{B} \cdot \vec{n}$

$\frac{\vec{B}_{T_1}}{\mu_1} = \frac{\vec{B}_{T_2}}{\mu_2}$

$$
\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\partial\mathbf{B}/\partial t \\
\nabla \times \mathbf{H} &= \mathbf{J}_c + \underbrace{\partial\mathbf{D}/\partial t}_{=0 \text{ when } \sigma = 0} \rightarrow J_D
\end{aligned}
$$

$J_d = \frac{J_c}{\frac{\sigma}{\omega\epsilon}}$

$$\mathbf{S_{avg}} = \frac{1}{2}Re\{\mathbf{E} \times \mathbf{H}^*\}$$



$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_v (\nabla \cdot \mathbf{F}) dv$$

$\oint \rightarrow \int_0^2 F(0)dz + \int_2^0 F(40)dz$

$$\oint_C \mathbf{F} \cdot d\ell = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

$$
\begin{aligned}
\oint_S \mathbf{D} \cdot d\mathbf{S} &= \int_v \rho\, dv \\
\oint_S \mathbf{B} \cdot d\mathbf{S} &= 0 \\
\oint_C \mathbf{H} \cdot d\ell &= \int_S \left(\mathbf{J}_c + \frac{\partial\mathbf{D}}{\partial t}\right) \cdot d\mathbf{S} \\
\oint_C \mathbf{E} \cdot d\ell &= \int_S -\frac{\partial\mathbf{B}}{\partial t} \cdot d\mathbf{S}
\end{aligned}
$$

$$\epsilon_0 = 8.85 \times 10^{-12} \quad \text{F/m}$$

$$\mu_0 = 4\pi \times 10^{-7} \quad \text{H/m}$$

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi\,\Omega$$

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \cdot \mathbf{H} &= 0 \\
\nabla \times \mathbf{E} &= -j\omega\mathbf{B} = -j\omega\mu\mathbf{H} \\
\nabla \times \mathbf{H} &= \mathbf{J}_c + j\omega\mathbf{D} = (\sigma + j\omega\epsilon)\mathbf{E}
\end{aligned}
$$

$$\partial^2 E_x/\partial z^2 + k^2 E_x = 0$$

$$\underbrace{v}_{3\times10^8} = \frac{\omega}{\beta} = f\lambda = \frac{1}{\sqrt{\epsilon\mu}}$$

m or cm

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$$

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right)}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right)}$$

$$
\begin{aligned}
E_x &= A\, e^{-jkz} + B\, e^{+jkz} \text{ V/m} \\
H_y &= \frac{1}{\eta_0}\left(A\, e^{-jkz} + B\, e^{+jkz}\right) \text{ A/m.}
\end{aligned}
$$

$$\frac{E_0^r}{E_0^i} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}$$

$$\frac{H_0^r}{H_0^i} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}$$

$$\frac{E_0^t}{E_0^i} = \frac{2\eta_2}{\eta_1 + \eta_2}$$

$$\frac{H_0^t}{H_0^i} = \frac{2\eta_1}{\eta_1 + \eta_2}$$

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Chapter 9

Time-Varying Fields and Maxwell's Equations

W. H. Hayt, Jr. and J. A. Buck, *Engineering Electromagnetics*, 9th ed., McGraw-Hill, Chapters 9, 2019.

Annotation of these lecture notes is recommended and encouraged.

Introduction

- In previous courses, you were introduced to the manner in which time-varying electric and magnetic fields are coupled to each other, as described by Maxwell's equations.
- This module serves as a brief review of this material before we start discussing propagating waves on transmission lines and in unbounded media.
- Chapter Outline:
  - 9.1 - Faraday's Law
  - 9.2 - Displacement Current
  - 9.3 - Maxwell's Equations in Point Form
  - 9.4 - Maxwell's Equations in Integral Form
  - 9.5 - The Retarded Potentials

Pedagogy

Read (several times in various sequences):

- the Chapter Brief – for Performance and Enabling Objectives and for context and motivation
- the Chapter Lecture Notes – for additional insights
- the Chapter - use SQ3R to enhance your efficiency
- the Chapter Supplement – for additional or clarifying material
- the Chapter Review Questions – to self-assess

Solve:

- the Example Problems (and Solutions) – to become acquainted with the subtleties of actual problems
- The Take Home Assignment – to practice problem solving without reference

Your goal for phase 1 is to be able to correctly interpret and apply the key formulas in obvious ways.

KNOWLEDGE

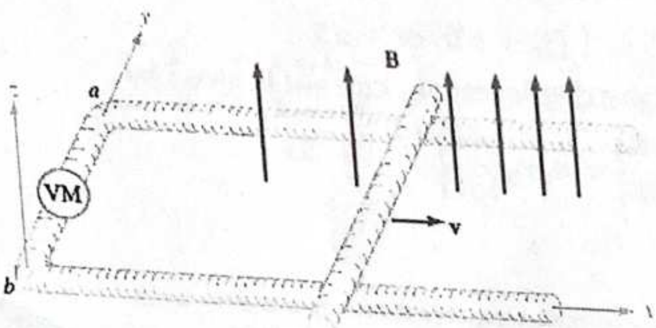
⇳

INTUITION

Your goal for phase 2 is to be able to correctly interpret and apply the key formulas in non obvious ways.

PO 1 - Faraday's Law

Give Faraday's law in customary, integral, and point form, and determine the EMF associated with a time-varying magnetic field, a time-constant flux, & a moving closed path. (§9.1)



$$\Phi = \int \mathbf{B} \cdot d\mathbf{S}$$

The relevant directions of circulation and the normal are defined by the right-hand rule

Recall

- the customary form: $emf = -N\, d\Phi/dt$
- the integral form: $emf = \oint \mathbf{E} \cdot d\mathbf{L} = -d/dt \int \mathbf{B} \cdot d\mathbf{S}$
- the point form: $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$
- that force on a charge is given by $\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$
- that the motional electric field intensity is given by $\mathbf{E}_m = \mathbf{v} \times \mathbf{B}$
- that including both the transformer and the motional emf yields

$$emf = \oint \mathbf{E} \cdot d\mathbf{L} = -\int \partial \mathbf{B}/\partial t \cdot d\mathbf{S} + \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L}$$

Note that a complete description of the above formulas requires a definition and the units of each parameter, and a sketch that describes the relevant geometry or scenario.

PO 2 - Time-variation Version of Ampère's Law

Give expressions for the time-variation version of Ampère's law in both point and integral form and calculate the magnitude of the displacement current in practical scenarios. (§9.2)

- Recall the point form, $\nabla \times \mathbf{H} = \mathbf{J}_c + \partial \mathbf{D}/\partial t$
- Recall the integral form, $\oint \mathbf{H} \cdot d\mathbf{L} = \int (\mathbf{J}_c + \partial \mathbf{D}/\partial t) \cdot d\mathbf{S}$
- Recognize that the magnitude of the displacement current is given by

$$|J_D| = \left|\frac{\partial \mathbf{D}}{\partial t}\right| = \epsilon_r \epsilon_0 \left|\frac{\partial \mathbf{E}}{\partial t}\right|$$

Note that a complete description of the above formulas requires a definition and the units of each parameter, and a sketch that describes the relevant geometry or scenario.

PO 3 - Maxwell's Equations

Give Maxwell's equations in point and integral form and the constitutive relations or auxiliary equations, explain the significance of the Helmholtz theorem to these results, and describe the relationship between the SI and previous systems of electromagnetic units. (§9.3 and §9.4)

- The Helmholtz theorem (or the fundamental theorem of vector calculus) tells us that any vector field can be resolved into irrotational and rotational components with former completely specified by the divergence of the field and the latter completely specified by the curl of the field.
- The general form of Maxwell's equations in point form is therefore predictable and inevitable with the right-hand side simply being experimental observations.

$$
\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\partial \mathbf{B}/\partial t \\
\nabla \times \mathbf{H} &= \mathbf{J}_c + \partial \mathbf{D}/\partial t
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{D} &= \epsilon_r \epsilon_0 \mathbf{E} \\
\mathbf{B} &= \mu_r \mu_0 \mathbf{H} \\
\mathbf{J} &= \sigma \mathbf{E}
\end{aligned}
$$

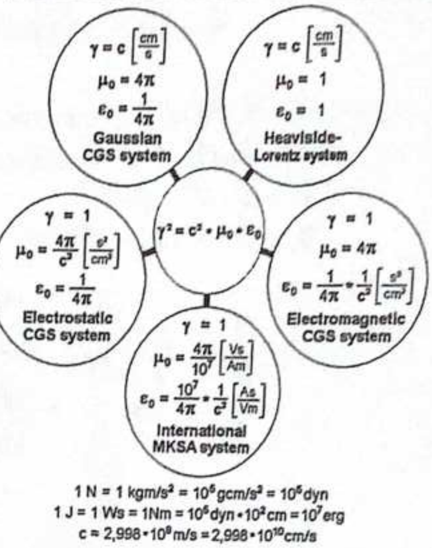
- The point forms are perhaps easier to visualize but assume continuous derivatives and cannot be used to analyze transitions across boundaries.
- The constitutive relations relate field quantities concerning flux to other field quantities concerning electric or magnetic fields.

$$
\begin{aligned}
\oint_S \mathbf{D} \cdot d\mathbf{S} &= \int_v \rho \, dv \\
\oint_S \mathbf{B} \cdot d\mathbf{S} &= 0 \\
\oint_C \mathbf{H} \cdot d\ell &= \int_S \left(\mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} \\
\oint_C \mathbf{E} \cdot d\ell &= \int_S -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}
\end{aligned}
$$

- Stokes' Theorem and the Divergence Theorem allow us to transform between the point and integral forms of Maxwell's equations.
- The integral forms are more general than the point forms because they do not assume continuous derivatives.
- As a result, the integral forms can be used to analyze transitions across boundaries.

- SI (formerly known as rationalized MKS) units are based on the fundamental units of metres, kilograms and seconds while the various CGS systems of units are based on the fundamental units of centimetres, grams and seconds.
- Conversion between CGS and SI units is not straightforward due to fundamental differences in their formulation.
- In most cases, referring to CGS implies CGS Gaussian.
- SI units are predominantly used in engineering applications and physics education, while Gaussian CGS units are commonly used in theoretical physics, descriptions of microscopic systems, relativistic electrodynamics, and astrophysics.

Electromagnetic systems of units



PO 4 - Boundary Conditions

Demonstrate the relationship between Maxwell's equations in point and integral form, and the boundary conditions imposed on field strength and flux density across material boundaries. (§9.3 and §9.4)

- Recognize that:
  - the *divergence theorem* links the divergence equations in **D** and **B**, *i.e.,* the flux densities, to the corresponding closed surface integrals.
  - these closed surface integrals can be used to deduce the continuity of the *normal component of the flux density* across a material interface.
  - *Stokes' theorem* links the curl equations in **E** and **H**, *i.e.,* the field strengths, to the corresponding closed line integrals
  - these closed surface integrals can be used to deduce the continuity of the *tangential component of the field strength* across a material interface.

We will see the details in the example problems.

Example Problems – A two-phase approach

Phase One – Understand the problem (intuition)

- Draw sketch(es)
- Identify key formulas or relationships
- Devise and communicate a strategy for obtaining a desired quantity in light of the given quantities

Phase Two – Solve the Problem (execution)

- Execute the strategy
  - derive,
  - substitute,
  - calculate
- Communicate the answer.

2.

Pedagogy

Read (several times in various sequences):

- the Chapter Brief – for Performance and Enabling Objectives and for context and motivation
- the Chapter Lecture Notes – for additional insights
- the Chapter - use SQ3R to enhance your efficiency
- the Chapter Supplement – for additional or clarifying material
- the Chapter Review Questions – to self-assess

Your goal for this phase is to be able to correctly interpret and apply the key formulas in obvious ways.

KNOWLEDGE

⇕

INTUITION

Solve:

- the Example Problems (and Solutions) – to become acquainted with the subtleties of actual problems
- The Take Home Assignment – to practice problem solving without reference

Your goal for this phase is to be able to correctly interpret and apply the key formulas in non obvious ways.

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ELEC 311 – Electromagnetic Fields and Waves

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Chapter 9 – Time Varying Fields and Maxwell's Equations

What you need to know!

A compilation of course performance objectives with detailed enabling objectives.

Where formulas are cited, be certain that you can identify each quantity and its units and sketch figures that describe the scenario.

1. Give Faraday's law in customary, integral, and point form, and determine the EMF associated with a time-varying magnetic field, a time-constant flux, & a moving closed path. (§9.1)
   - Recall the customary form: $emf = -N\, d\Phi/dt$
   - Recall the integral form: $emf = \oint \mathbf{E}\cdot d\mathbf{L} = -d/dt \int \mathbf{B}\cdot d\mathbf{S}$
   - Recall the point form: $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$
   - Recall that force on a charge is given by $\mathbf{F} = Q\mathbf{v}\times\mathbf{B}$
   - Recall that the motional electric field intensity is given by $\mathbf{E}_m = \mathbf{v}\times\mathbf{B}$
   - Recall that including both the transformer and the motional emf yields

$$emf = \oint \mathbf{E}\cdot d\mathbf{L} = -\int \partial\mathbf{B}/\partial t\cdot d\mathbf{S} + \oint(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{L}$$

2. Give expressions for the time-variation version of Ampère's law in both point and integral form and calculate the magnitude of the displacement current in practical scenarios. (§9.2)
   - Recall the point form, $\nabla\times\mathbf{H} = \mathbf{J}_c + \partial\mathbf{D}/\partial t$
   - Recall the integral form, $\oint \mathbf{H}\cdot d\mathbf{L} = \int(\mathbf{J}_c + \partial\mathbf{D}/\partial t)\cdot d\mathbf{S}$
   - Recognize that the magnitude of the displacement current is given by

$$|J_D| = \left|\frac{\partial\mathbf{D}}{\partial t}\right| = \epsilon_r\epsilon_0\left|\frac{\partial\mathbf{E}}{\partial t}\right|$$

3. Give Maxwell's equations in point and integral form and the constitutive relations or auxiliary equations, explain the significance of the Helmholtz theorem to these results, and describe the relationship between the SI and previous systems of electromagnetic units. (§9.3 and §9.4)

$$\begin{aligned}
\nabla\cdot\mathrm{D} &= \rho \\
\nabla\cdot\mathrm{B} &= 0 \\
\nabla\times\mathrm{E} &= -\partial\mathrm{B}/\partial t \\
\nabla\times\mathrm{H} &= \mathrm{J}_c + \partial\mathrm{D}/\partial t
\end{aligned}
\qquad\qquad
\begin{aligned}
\mathrm{D} &= \epsilon_r\epsilon_0\mathrm{E} \\
\mathrm{B} &= \mu_r\mu_0\mathrm{H} \\
\mathrm{J} &= \sigma\mathrm{E}
\end{aligned}$$

$$
\begin{aligned}
\oint_S \mathbf{D} \cdot d\mathbf{S} &= \int_v \rho \, dv \\
\oint_S \mathbf{B} \cdot d\mathbf{S} &= 0 \\
\oint_C \mathbf{H} \cdot d\ell &= \int_S \left(\mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} \\
\oint_C \mathbf{E} \cdot d\ell &= \int_S -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}
\end{aligned}
$$

- The Helmholtz theorem (or the fundamental theorem of vector calculus) tells us that any vector field can be resolved into irrotational and rotational components with former completely specified by the divergence of the field and the latter completely specified by the curl of the field.
- The general form of Maxwell's equations in point form is therefore predictable and inevitable with the right-hand side simply being experimental observations.
- SI (formerly known as rationalized mks) units are based on the fundamental units of metres, kilograms and seconds while the various cgs systems of units (Gaussian, electrostatic, electromagnetic and Lorentz-Heaviside) are based on the fundamental units of centimetres, grams and seconds.
- Conversion between cgs and SI units is not straightforward due to fundamental differences in their formulation.

4. Demonstrate the relationship between Maxwell's equations in point and integral form, and the boundary conditions imposed on field strength and flux density across material boundaries. (§9.3 and §9.4)
    - Recognize that the *divergence theorem* links the divergence equations in **D** and **B**, *i.e.*, the flux densities, to the corresponding closed surface integrals.
    - Recognize that these closed surface integrals can be used to deduce the continuity of the *normal component of the flux density* across a material interface.
    - Recognize that *Stokes' theorem* links the curl equations in **E** and **H**, *i.e.*, the field strengths, to the corresponding closed line integrals
    - Recognize that these closed surface integrals can be used to deduce the continuity of the *tangential component of the field strength* across a material interface.

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ELEC 311 – Electromagnetic Fields and Waves

Fall 2022

Recitation Questions for

Chapter 9 – Time-Varying Fields and Maxwell's Equations

in W. H. Hayt, Jr. and J. A. Buck, *Engineering Electromagnetics*, 9th ed., McGraw-Hill, 2019, pp. 279-302.

The purpose of these recitation questions is to assist the reader in assessing their mastery of the key concepts introduced in this chapter. The answers can be found in the textbook. ELEC 311 students should be prepared to provide answers to these questions in class or on an exam.

Introduction

1. What is the focus of this chapter?
2. How did Maxwell and Faraday know each other?

9.1 Faraday's Law

1. What inspired Faraday to believe that a magnetic field could induce a current?

9.1.1 Faraday's Law in Point and Integral Form

1. How is Faraday's law customarily stated?
2. What three situations may result in a non-zero value of $d\Phi/dt$?
3. What is the significance of the minus sign in Faraday's law?
4. What is Lenz's law?
5. How is Faraday's law modified to account for an *N*-turn filamentary conductor, i.e., an *N*-turn thin wire coil?
6. What is emf and how is it defined?
7. How can Φ be defined in terms of B?
8. What theorem relates the integral form of Faraday's law in (4) to the point form in (6)?
9. Explain the right-hand relationship between the integrals in (5).
10. How do (4) and (6) compare when fields are: a) time-varying and b) static?

9.1.2 EMF Arising from a Time-Varying Magnetic Field

1. Briefly describe the scenario described in this section
2. What caution do the authors offer concerning the given field **B**?

9.1.3 Motional EMF

1. Briefly describe the scenario described in this section.
2. How is motional emf defined?
3. To what scenario does (14) correspond? Explain the significance of each term.

$$emf = \oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L} \qquad (14)$$

4. What types of contrived examples may present difficulties when one attempts to apply Faraday's law?
5. How is the separation of emf into two parts in (14), one due to the time rate of change of **B** and the other to the motion of the current, somewhat arbitrary?

9.2 Displacement Current

9.2.3 Modifying Ampère's Law for Time-Varying Fields

1. What is Ampère's law for steady magnetic fields?
2. What is the equation of continuity, also known as the continuity condition?
3. What paradox occurs if one takes the divergence of Ampère's law for steady magnetic fields?
4. How can this paradox be resolved?
5. What is Ampère's law for time-varying magnetic fields in point form?
6. What are three types of current density?
7. What is the result of applying Stokes' theorem to Ampère's law for time-varying magnetic fields in point form?

9.2.3 An Illustration of Displacement Current

1. Briefly describe the scenario described in this section.
2. What is an expression for the displacement current within the capacitor?
3. When does displacement current exist?
4. Why was displacement current likely never discovered experimentally?

9.3 *Maxwell's Equation in Point Form*

1. Recite the four Maxwell's equations for time-varying fields in point form. State the physical significance of each.

2. What is the most important implication of (20) that distinguishes time-varying from static scenarios?

3. State the four auxiliary equations or constitutive relations. State the physical significance of each.

4. What are the corresponding relationships involving the polarization and magnetization fields?

5. What is the Lorentz force equation?

9.4 *Maxwell's Equations in Integral Form*

1. Recite the four Maxwell's equations for time-varying fields in integral form. State the physical significance of each.

2. How can Maxwell's equations for time-varying fields in integral form be obtained from the corresponding equations in point form?

3. How do Maxwell's equations for time-varying fields in integral form allow us to find the conditions for **B**, **D**, **H**, and **E** at media boundaries, *i.e.,* boundary conditions?

4. What are the conditions for **B**, **D**, **H**, and **E** at media boundaries with and without charges or currents on the boundaries?

5. Why is it necessary to account for the conditions for field strength and flux density at media boundaries?

9.5 *The Retarded Potentials*

1. Give expressions for the scalar electric potential in both integral and point form under static conditions?

2. Give expressions for the vector magnetic potential in both integral and point form under dc conditions?

3. How does one obtain **E** and **B** from V and **A** under static or dc conditions?

4. How must the expression for **E** in (49) be modified when fields are time-varying?

5. How must the expression for **B** in (50) be modified when fields are time-varying?

6. What form does the expression for V in (45) take when fields are time-varying?

7. What form does the expression for **A** in (46) when fields are time-varying?

8. Why are the expressions for V and **A** in (45) and (46) referred to as retarded potentials?

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ELEC 311 – Electromagnetic Fields and Waves

Fall 2022

A Supplement to

Chapter 9 – Time-Varying Fields and Maxwell's Equations

in W. H. Hayt, Jr. and J. A. Buck, *Engineering Electromagnetics*, 9th ed., McGraw-Hill, 2019, pp. 279-302.

The purposes of this supplement are:

- to assist the reader in identifying key points to be recognized as they apply the SQ3R (Survey, Question, Read, Recite, Review) process to reading and reviewing the chapter and
- to provide comments and supplemental information that fill in apparent gaps in the textbook.

Introduction

For the case of *static* fields, Maxwell's equations in point form reduce to

$$\begin{aligned} \nabla \cdot \boldsymbol{D} &= \rho_v \\ \nabla \cdot \boldsymbol{B} &= 0 \\ \nabla \times \boldsymbol{E} &= 0 \\ \nabla \times \boldsymbol{H} &= \boldsymbol{J} \end{aligned}$$

and, when observed from a fixed point in space, *i.e.,* a static observer, the electric and magnetic fields exist independently of each other.

For the case of time-varying fields, the electric and magnetic fields are coupled and the above description needs to be revised. Such coupling between the electric and magnetic fields allows for the possibility of propagating electromagnetic waves which travel at the speed of light in a medium.

The observations by: 1) Oersted in 1820 that a fixed current (associated, of course, with a static electric field) gives rise to a magnetic field and 2) Faraday in 1831 that a time-varying magnetic field gives rise to an electric field established the link between electric and magnetic fields and the notion that electromagnetism is a single unified phenomenon. However, the descriptions of the curl of the magnetic and electric fields given by Ampère's Law and Faraday's Law, respectively, did not completely capture the full relationship between time-varying electric and magnetic fields.

In 1860, Maxwell resolved the continuity-of-current paradox by identifying displacement current as the missing component on the right-hand side of Ampere's Law. With a complete description of the electromagnetics in hand, Maxwell was able to predict the existence of electromagnetic waves that propagate at the speed of light and thereby establish light as an electromagnetic wave and thereby confirm Faraday's speculation of decades earlier.

We quickly note, however, Maxwell expressed his work in the form of quaternions, a mathematical formulation originally devised by Hamilton in 1843 for use in classical mechanics. The modern vector calculus notation that you became familiar with in ELEC 211 and which you will use here is due to the work of Oliver Heaviside, Josiah Gibbs, and others (the Maxwellians) in the late 1800's.

The remainder of this *chapter* is concerned with the complete description of the electromagnetic field that was first described by James Clerk Maxwell and rendered in their modern form by Oliver Heaviside and Josiah Gibbs.

The remainder of this *course* is concerned with the practical manifestation of propagating electromagnetic waves along guided structures and in unbounded media.

9.1 *Faraday's Law*

The section:

- provides the historical context for the revelation of Faraday's law
- defines emf and magnetic flux and gives Faraday's law in customary, point and integral form
- considers how emf may result from:
  - a time-varying magnetic field
  - a time-constant flux and a moving closed path
  - a time-varying magnetic field *and* a moving closed path
  - contrived examples where application of Faraday's law may be difficult

Comments

The descriptions are very complete but much more could be said regarding the application of special relativity to electromagnetics and the consistency of Maxwell's equations with special relativity. Indeed, when Einstein was once asked if he stood on the shoulders of Newton, he replied "No, on the shoulders of Maxwell."[1]

Suppose a fixed observer sees only a static electric field. An observer moving with respect to the charges or potentials that give rise to the electric field will see both a static electric field and a static magnetic field! Similarly, where a fixed observer sees only a static magnetic field, an observer moving with respect to the current that gives rise to the magnetic field will also see both a static electric field and a static magnetic field!

The details of the *Lorentz transformation of the electric and magnetic fields between different rest frames* are a consequence (or predictor?) of special relativity but are generally not considered in engineering electromagnetic courses. However, engineering students should at least be aware of this phenomenon which can have important engineering consequences beyond the terse comments in the last paragraph of p. 285.

A case in point is the TRIUMF cyclotron on the UBC south campus, http://www.triumf.ca. The 520 MeV cyclotron accelerates negative hydrogen ions to about 75% of the speed of light. In their rest frame, the fast-moving negative hydrogen ions perceive the static magnetic field that guides their spiral trajectory through the cyclotron as having both magnetic and electric field components. If the electric field component is too strong, the negative hydrogen ion will lose its extra electron and become a neutral atom that would no longer be guided by the magnetic field.

As a result, the designers of the 520 MeV cyclotron were forced to use a fairly weak magnetic field in order to reduce the strength of the static electric field experienced by negative hydrogen ions

[1] https://www.theguardian.com/science/2015/dec/08/einstein-inspired-by-james-clerk-maxwell

travelling at $0.75c$. This results in the ions having a fairly large orbital radius and is the reason why the TRIUMF 520 MeV cyclotron is so physically large compared to cyclotrons with similar energies that accelerate protons instead, *e.g.*, the 590 MeV cyclotron at the Paul Scherrer Institut in Switzerland.

9.2 *Displacement Current*

This section:

- considers the limitations of Ampère's law when fields are time-varying
- the manner in which the introduction of displacement current resolves the issue
- expressions for the time-variation version of Ampère's law in both point and integral form
- an illustration of displacement current within a parallel-plate capacitor

Comments

The descriptions are very complete.

9.3 *Maxwell's Equation in Point Form*

This section:

- gives the four Maxwell's equations in point form
- considers the physical implications of the equations as stated
- gives the auxiliary equations, also known as the constitutive relations
- gives the relationships between field strength and flux density involving the polarization and magnetization fields
- gives the Lorentz force equation

Comments

The descriptions are very complete. However, the notion that the electric and magnetic fields are each completely described by their divergence and curl is not peculiar to electromagnetics but is instead a consequence of Helmholtz's theorem which states that any vector field A can be expressed as the sum of irrotational and rotational components where:

- the irrotational component is completely described by the divergence of A, and,
- the rotational component is completely described by the curl of A.

Accordingly, there must be four equations in Maxwell's set as shown above to completely describe the electric and magnetic fields.

Having said that, the expressions on the right-hand side of each of the divergence and curl equations cannot be derived from first principles and are simply experimental observations. For example, there is no reason that $\nabla \cdot \boldsymbol{B}$ must be equal to zero except that magnetic charges have never been observed. (The first person to experimentally demonstrate the existence of free magnetic charges will rewrite electromagnetics textbooks and almost certainly win the Nobel Prize.)

9.4 *Maxwell's Equations in Integral Form*

This section:

- gives the four Maxwell's equations in integral form
- notes that Maxwell's equations in integral form can be used to derive the boundary conditions for electromagnetic field quantities at interfaces between material media
- explains why it is practically imperative to know the boundary conditions for electromagnetic field quantities at interfaces between material media

Comments

The descriptions are fairly complete but the details of the derivation of the boundary conditions for flux density and field strength at material boundaries are strangely absent. We will resolve this in the problem set solutions.

9.5 *The Retarded Potentials*

This section:

- gives the scalar electric and magnetic vector potentials for the static/dc case
- shows how they can be used to obtain the fundamental fields
- shows how the expression for $\mathbf{E}$ in terms of V in the static case is inadequate in the time-varying case and shows how it can be resolved by adding a new term, $-\partial\mathbf{A}/\partial t$
- introduces the retarded scalar electric and magnetic vector potentials

Comments

The descriptions are very complete and will play a key role in solving the problem of radiation from a current element in ELEC 411 – Antennas and Propagation.

References

In addition to the seven references given at the end of the chapter, ELEC 311 students may also wish to consult the following:

[1] R. F. Harrington, *Introduction to Electromagnetic Engineering*, McGraw-Hill, 1958; Dover, 2003.

[2] R. F. Harrington, *Time-Harmonic Electromagnetic Fields*, McGraw-Hill, 1961; Wiley – IEEE Press, 2001.

These two books are written in a similar style with the former aimed at undergraduate students and the latter aimed at graduate students. Both books are renowned for their clarity and style. *Time-Harmonic Electromagnetic Fields* is widely regarded as a classic work in this field and was re-issued by IEEE Press in 2001. *Introduction to Electromagnetic Engineering* was re-issued by Dover in 2003.

[3] R. Feynmann, *The Feynman Lectures on Physics, Vol. II: Mainly Electromagnetism and Matter,"* Addison Wesley, 1964; Basic Books, 2011.

Told from a physics perspective, Feynmann's lectures have attracted considerable praise for their unique combination of style and substance. Originally published in 1964, the three-volume set was reissued as a Millennium Edition by Basic Books in 2011.

[4] D. Fleisch, *A Student's Guide to Maxwell's Equations,* Cambridge University Press, 2008.

This book and its companion website at http://www.danfleisch.com/maxwell/ have attracted an enormous following and very strong reviews from readers.

On the book's website, readers will find:

Complete solutions to every problem in the book

You can get a series of hints to help you solve the problem, or you can see the full solution straight away. Just use the menu on the left to click on one of the Chapters, select "Problems," and pick the problem you want to work on.

Audio podcasts

For every module in the chapters covering the four Maxwell's Equations, Fleisch will walk you through the major concepts contained within that module. These audio files can be streamed to your computer so you can hear them immediately, or you can use your favorite podcast-catching software to grab and store them.

Three-dimensional models of electric and magnetic fields

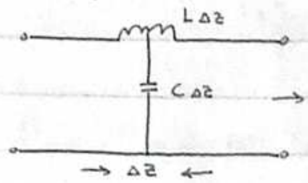
The twelve sketches of electric and magnetic fields shown in Figures 1.1 and 2.1 in the book are available on this site as VRML files so you can see the fields in three dimensions. You'll need a VRML plug-in for your browser to view these files – you can find several free 3D viewers on the Web (Fleisch has been using Cortona® with good success).

Review of rectangular, cylindrical, and spherical coordinates

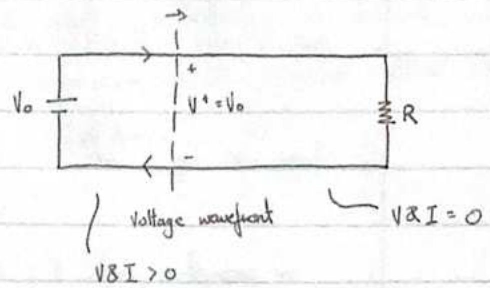
To keep the book short (and inexpensive), Fleisch didn't include a review of coordinate systems in the text – so it's here on the website. If you're a bit fuzzy on how to get from from (x,y,z) to (r, theta, phi), or from (A_x, A_y, A_z) to (A_r, A_{theta}, A_{phi}), then you should take a look at this review, which is in .pdf format.

Chapter 10 - Transmission Lines

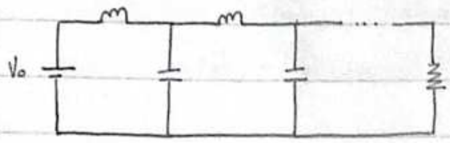
Basic Concepts



$\rightarrow z$ $\quad C\,[F/m] \quad L\,[H/m]$



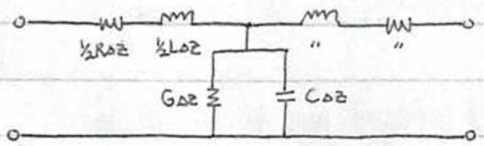
Discrete Element Model



series L & shunt C creates time delay.

at SS, L → wire C → battery with V_0 , $I_R = \frac{V_0}{R}$

Line Model with Loss



conductance

$R\,[\Omega/m] \quad G\,[S/m]$

L, C, G, R comprise the primary constants of transmission line.

Using KCL, $\frac{\partial V}{\partial z} = -(RI + L\frac{\partial I}{\partial t})$ $\quad \frac{\partial I}{\partial z} = -(GV + C\frac{\partial V}{\partial t})$ — Telegraphist's equation.

Differentiating eq's, $\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2} + (LG + RC)\frac{\partial V}{\partial t} + RGV$ $\quad \frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2} + (LG + RC)\frac{\partial I}{\partial t} + RGI$

voltage wave eq $\qquad$ current wave eq

Zero R & G gives a lossless line, $\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}$. This eq has sol'n $V(z,t) = \underbrace{f_1(t - \frac{z}{v})}_{bak} + \underbrace{f_2(t + \frac{z}{v})}_{fwd}$

lossless line eq $\qquad$ velocity parameter

$= V^+ + V^-$

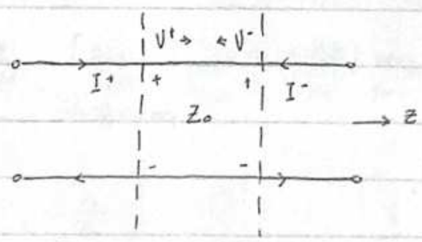
Propagation velocity, $v = \frac{1}{\sqrt{LC}}$

$I(z,t) = \frac{1}{Lv}[V^+ - V^-]$

$= I^+ + I^-$

Characteristic Impedance, $Z_0 = Lv = \sqrt{\frac{L}{C}}$

Also, $V^+ = Z_0 I^+ \quad V^- = -Z_0 I^-$



$\lambda = \frac{2\pi}{\beta} = \frac{v_p}{f}$

$V_f(z,t) = |V_0|\cos(\omega t - \beta z)$ $\quad$ where phase constant $\beta = \frac{\omega}{v_p}$

$V_b(z,t) = |V_0|\cos(\omega t + \beta z)$ $\quad$ phase velocity

Voltage Wave in CMPLX Form

phasor voltage wave, $V_s(z) = V_0 e^{\pm\beta z j}$

inst. V wave, $V_c(z,t) = V_0 e^{\pm\beta z j} e^{j\omega t}$ in real form $= \frac{1}{2} V_s(z) e^{j\omega t} + c$

phasor wave eq, $\frac{\partial^2 V_s}{\partial z^2} = \underbrace{(R+j\omega L)}_{\substack{\text{series impd} \\ (Z)}}\underbrace{(G+j\omega C)}_{\substack{\text{shunt admittance} \\ (G)}} V_s = \gamma^2 V_s$ with sol'n $V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$

line propagation const. $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{ZY} = \underbrace{\alpha}_{\text{attenuation}} + j\underbrace{\beta}_{\text{phase}}$

Telegraphist's, $\frac{dV_s}{dz} = -(R+j\omega L) I_s = -ZI_s$ $\frac{dI_s}{dz} = -(G+j\omega L) V_s = -YV_s$

Charateristic Impedance, $Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{(R+j\omega L)}{(G+j\omega C)}} = |Z_0| e^{j\theta}$

Low - Loss

$G \ll \omega C$ & $R \ll \omega L$

$\alpha \doteq \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)$ $\beta \doteq \omega\sqrt{LC}\left[1 + \frac{1}{8}\left(\frac{G}{\omega C} - \frac{R}{\omega L}\right)^2\right]$

$Z_0(G=0) \doteq \sqrt{\frac{L}{C}}\left(1 - \frac{R}{2\omega L} j\right) = |Z_0| e^{j\theta}$ $\tan^{-1}\left(\frac{-R}{2\omega L}\right)$

$\sqrt{\frac{L}{C}}$

Power Transmission

$P(z,t) = |V_0||Z_0| e^{-\alpha z} \cos(\omega t - \beta z)\cos(\omega t - \beta z + \theta)$

$P_{avg}(z,t) = \frac{1}{T}\int_0^T P(z,t)$

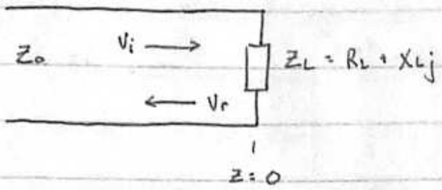
$= \frac{1}{2} \mathrm{Re}\{V_s I_s^*\}$

Power loss in Decibel

$\langle P(z)\rangle = \langle P(0)\rangle e^{-2\alpha z}$

Power loss [dB] $= 10\log_{10}\underbrace{\left[\frac{\langle P(0)\rangle}{\langle P(z)\rangle}\right]}_{\text{power fraction}} = 8.69\alpha z$ where $\alpha = \frac{PL}{8.69 z}$

Reflections



The diff of Z_o creates contradiction in the line.
Backward propagating wave or the reflection voltage (V_r) form due to this contradiction.

Voltage Waves

$$\left.\begin{aligned} V_i(z) &= V_{oi} e^{-\alpha z} e^{-j\beta z}, \quad V_i(0) = V_{oi} \\ V_r(z) &= V_{or} e^{+\alpha z} e^{+j\beta z} \quad V_r(0) = V_{or} \end{aligned}\right\} z = 0, \text{ at the load}$$

Boundary Conditions at Load

$V_L = V_{oi} + V_{or}$

$I_L = I_{oi} + I_{or}$

From this we can get the ratio of V_{or} & V_{oi}, gamma (Γ), reflection coeff.

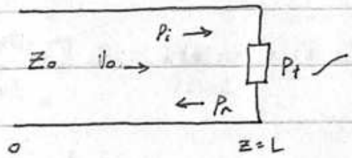
$$\Gamma = \frac{V_{or}}{V_{oi}} = \frac{Z_L - Z_o}{Z_L + Z_o} = |\Gamma| e^{j\phi_r}$$

Using Γ, we can define the ratio of V_L & V_{oi}, tau (τ), transmission coeff.

$$\tau = \frac{V_L}{V_{oi}} = 1 + \Gamma = \frac{2Z_L}{Z_o + Z_L} = |\tau| e^{j\phi_t}$$

if $\Gamma > 0 \rightarrow \tau > 1 \rightarrow V_L > V_{oi}$ which seem strange but... $I_L < I_{oi} \rightarrow P_L < P_i$

Power



dissipated power by the load.

$P = \frac{1}{2} \mathrm{Re}\{ V_s I_s^* \}$ $\quad P_i = \frac{1}{2} \frac{|V_o|^2}{|Z_o|} e^{-2\alpha L} \cos\theta$ $\quad P_r = \frac{1}{2} \frac{|\Gamma|^2 |V_o|^2}{|Z_o|} e^{-2\alpha L} \cos\theta$

From this, $\frac{P_r}{P_i} = \Gamma \Gamma^* = |\Gamma|^2$ & $\frac{P_t}{P_i} = 1 - |\Gamma|^2$

Line Total Voltage

$$V_{sT}(z) = V_0 e^{-j\beta z} + \Gamma V_0 e^{j\beta z}$$
$$= V_0 (1 - |\Gamma|) e^{-j\beta z} + 2V_0 |\Gamma| e^{j\phi/2} \cos(\beta z + \phi/2)$$

In instantaneous real form

$$V(z,t) = \mathrm{Re}[V_{sT}(z) e^{j\omega t}] = \underbrace{V_0 (1 - |\Gamma|) \cos(\omega t - \beta z)}_{\text{traveling wave}} + \underbrace{2|\Gamma| V_0 \cos(\beta z + \phi/2) \cos(\omega t + \phi/2)}_{\text{standing wave}}$$

Min. Voltage Location

$$z_{min} = -\frac{1}{2\beta}(\phi + (2m+1)\pi) \quad \text{where } m = 0, 1, 2 \ldots$$

Using this, we can find $V_{sT}(z_{min}) = V_0 e^{j\phi/2} e^{j\pi(2m+1)/2} (1 + |\Gamma| e^{-j(2m+1)\pi})$

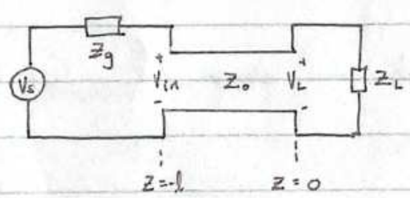
$$|V_{sT}(z_{min})| = V_0 (1 - |\Gamma|)$$

similarly $|V_{sT}(z_{max})| = V_0 (1 + |\Gamma|)$ where $z_{max} = \frac{-1}{2\beta}(\phi + 2m\pi)$

The ratio of $V_{sT}(z_{max})$ & $V_{sT}(z_{min})$, voltage standing wave ratio (VSWR), s.

$$s = \frac{V_{sT}(z_{max})}{V_{sT}(z_{min})} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \rightarrow |\Gamma| = \frac{s-1}{s+1}$$

Finite Length Line



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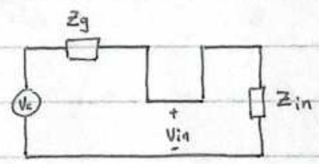


The power dissipated by the input impedance = P_t

$$V_{sT}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$
$$I_{sT}(z) = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z}$$

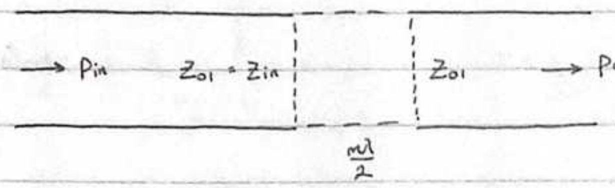
with wave impedance $Z_w(z) = \frac{V_{sT}(z)}{I_{sT}(z)} = Z_0 \left[\frac{e^{-j\beta z} + \Gamma e^{j\beta z}}{e^{-j\beta z} - \Gamma e^{j\beta z}}\right]$

$$= Z_0 \left[\frac{Z_L \cos(\beta z) - jZ_0 \sin(\beta z)}{Z_0 \cos(\beta z) - jZ_L \sin(\beta z)}\right]$$

$$Z_{in} = Z_w(z = -l)$$
$$= Z_0 \left[\frac{Z_L \cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + jZ_L \sin(\beta l)}\right]$$

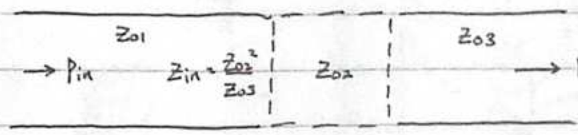
Special Cases

Half-wave, $Z_{in}(l = \frac{n\lambda}{2}) = Z_L$ where $\beta l = n\pi$



$\Gamma = 0 = \frac{Z_{in} - Z_{01}}{Z_{in} + Z_{01}}$, same charac.-imp.

Quarter-wave, $Z_{in}(l = \lambda/4) = \frac{Z_0^2}{Z_L}$ where $\beta l = (2n+1)\frac{\pi}{2}$



$Z_{02} = (Z_{01} Z_{03})^{1/2}$, diff charac.-imp.

Smith chart

$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi} = \Gamma_r + j\Gamma_i$

$z_L = \frac{Z_L}{Z_0} = \frac{R_L + jX_L}{Z_0} = r + jx \rightarrow \Gamma = \frac{z_L - 1}{z_L + 1}$ & $z_L = \frac{1+\Gamma}{1-\Gamma}$ & $z_w(z) = \frac{Z_w(z)}{Z_0} = \frac{1 + \Gamma e^{2\beta z j}}{1 - \Gamma e^{2\beta z j}}$

normalize to get rid of Z_0

$z_L = z_w(z=0) = \frac{1+\Gamma}{1-\Gamma}$

normalising z_w changes Γ.

$\Gamma \Rightarrow \Gamma e^{2\beta z j}$

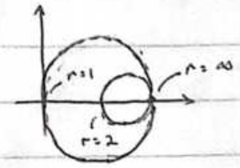
$\Gamma \Rightarrow |\Gamma| e^{j(\phi + 2\beta z)}$

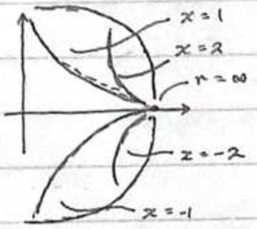
resistive

$r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}$

reactive

$x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i}$





Entire chart represents $\frac{\lambda}{2}$.

Chapter 11 - Uniform Plane Waves

Time Harmonic EM-Field:

We assume a time factor $e^{j\omega t}$

$\nabla \cdot \vec{E} = 0 \qquad \nabla \cdot \vec{H} = 0 \qquad \nabla \times \vec{E} = -j\omega \vec{B} = -j\omega\mu \vec{H} \qquad \nabla \times \vec{H} = J_c + j\omega \vec{D} = (\sigma + j\omega\varepsilon)\vec{E}$

With this we can get the wave equation,

$\nabla \times (\nabla \times \vec{H}) = (\sigma + j\omega\varepsilon)(\nabla \times \vec{E}) \qquad \nabla \times (\nabla \times \vec{E}) = -j\omega\mu(\nabla \times \vec{H})$

$\nabla \times \nabla \times \vec{A} = -\nabla^2 A$

$\nabla^2 \vec{H} = j\omega\mu(\sigma + j\omega\varepsilon)\vec{H} = \gamma^2 \vec{H} \qquad \nabla^2 \vec{E} = j\omega\mu(\sigma + j\omega\varepsilon)\vec{E} = \gamma^2 \vec{E}$

$\gamma = \alpha + j\beta$

$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)} \qquad \beta = \omega \sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)} \qquad$ with $v = \frac{\omega}{\beta} = f\lambda$

Plane-Wave Solution:

Plane-wave equation $\overset{\text{relate}}{-}$ lossless medium. (Helmholtz)

$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0 \qquad \frac{d^2 E_y}{dz^2} + k^2 E_y = 0 \qquad$ with $k = \omega\sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda} = \beta$

horizontal — vertical.

combining the two, we can get a general case.

Solutions for the equation

$E_x = Ae^{-jkz} + Be^{jkz}$ V/m $\qquad H_y = \frac{1}{\eta_0}(Ae^{-jkz} + Be^{jkz})$ A/m $\qquad e^{-jkz}$ +Z dir $\qquad e^{jkz}$ −Z dir

$\eta_0 \rightarrow \left|\frac{E_x}{H_y}\right|$

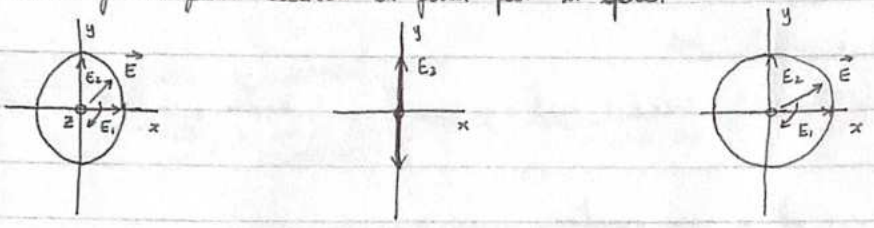
Power

$\vec{S} = \frac{1}{2}\mathrm{Re}\{\vec{E} \times \vec{H}^*\}$ W/m² with solution for $e^{-jkz} \rightarrow \vec{S} = \frac{A^2}{2\eta_0}\hat{z}$

$e^{jkz} \rightarrow \vec{S} = -\frac{B^2}{2\eta_0}\hat{z}$

Polarisation

Behavior of E-field observed at fixed-point in space.



general - case (Elliptical) (linear) special - case (circular)

Generalization of wave

$\frac{\partial^2 F}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2}$, scalar wave equation in 1D. with solution $F = f_1(z - vt)$ (arbitrary function)

$f_2(z + vt)$

Perfect Dielectric :

$\sigma = 0$, perfect dielectric. $\rightarrow$ $\alpha = 0$ $\beta = \omega\sqrt{\mu\varepsilon}$ $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

$\frac{\sigma}{\omega\varepsilon} < 0.1$, good dielectric

wave don't suffer α

$\vec{E}$ & $\vec{H}$ always $\perp$

$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$

Partially Conductive

$\vec{E} = E_0 e^{-\gamma z}\,\hat{x}$ $\nabla\times\vec{E} = -j\omega\mu\vec{H}$ where $\frac{E_x}{H_y}$ is intrinsic impedance

$\vec{H} = \left(\frac{\sigma + j\omega\varepsilon}{j\omega\mu}\right)^{1/2} E_0 e^{-\gamma z}\,\hat{y}$

$E(z,t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}\,\hat{x}$ $H(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} e^{j(\omega t - \beta z - \theta)}\,\hat{y}$

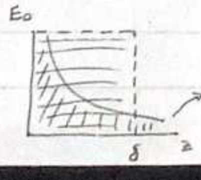
Good Conductor

$\sigma \gg \omega\varepsilon$, good conductor

$\gamma = \alpha + j\beta$ $\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma}$ $\eta = \sqrt{\frac{\omega\mu}{\sigma}}\ \angle 45°$

Skin Depth

$\delta = \frac{1}{\sqrt{\pi f\mu\sigma}} = \frac{1}{\alpha}$



$e^{-\alpha z}$

I

I flow 5 δ of skin depth

Spherical Waves :

Antennas radiate spherical waves.

Helmholtz equation in spherical coord.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial^2\psi}{\partial\phi^2}\right) + k^2\psi = 0$$

with solution $\psi = R(r)\,\Theta(\theta)\,\Phi(\phi)$

for simplest case, we use symmetry to ignore harmonics

$$\Phi(r) = C\frac{e^{-j\beta r}}{r} \qquad \Phi(r) = D\frac{e^{j\beta r}}{r}$$

Power and the Poynting Vector :

$$P(t) = \oint (\vec{E}\times\vec{H})\cdot d\vec{s} = \oint_s \vec{S}\cdot d\vec{s}$$, $\vec{S}$ is poynting vectors.

$$S_{avg} = \frac{1}{2}\,\mathrm{Re}\{\vec{E}\times\vec{H}^*\}$$

Chapter 12

In charge free condition $\vec{D}_{n_1} = \vec{D}_{n_2}$ & $\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r_2}}{\varepsilon_{r_1}}$

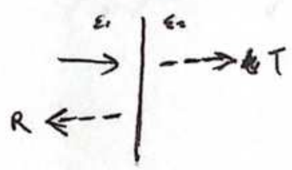
In all cases $E_{t_1} = E_{t_2}$

In current free cond. $H_{t_1} = H_{t_2}$

In all cases $B_{n_1} = B_{n_2}$

no current $\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_{r_2}}{\mu_{r_1}}$

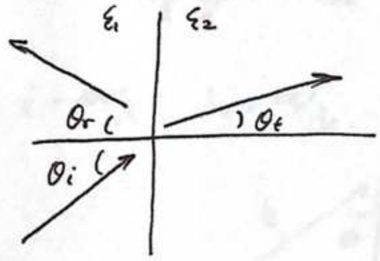
Normal Incidence



@ Boundary, $E_0^i + E_0^r = E_0^t$

$H_0^i - H_0^r = H_0^t$

Oblique Incidence & Snell's Laws

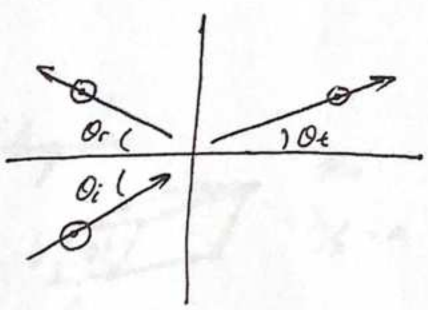


Snell Law:

$\theta_i = \theta_r$

$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}} = \frac{v_1}{v_2}$

Perpendicular or TE Polarization



E ⊥ plane of incidence

Parallel or TM Polarization

E is // to the plane of incidence

@ $\mu_1 = \mu_2$, there ~~are~~ is no reflected angle. $\theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$, Brewster angle.

* $\eta \neq n$

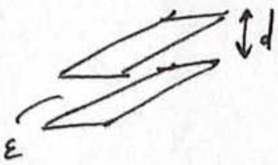
Ch 13

Uniform Plane Wave (TEM)

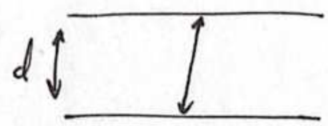
Have I & V $\qquad Z_0 = \sqrt{L/C}$

Traverse Magnetic (TM)

propagate through parallel plates



Traverse Electric (TE)

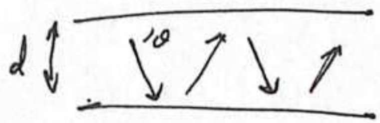


spacing = $\lambda/2$.

↳ perfect reflection.

but the wave is still TEM.

$\therefore d = \frac{m\lambda}{2}$



this is not TEM.

when $\theta = 0$ → perf. reflection.

m = # of half-cycle → cut-off

= # E-field maxima

• f_c is lowest that the structure can support.

change mode. $m \cdot \lambda_c = 2d \qquad f_c = \frac{v_p}{\lambda_c} \quad v = \frac{c}{n}$

Group V = c/n is the v in consequence of angle generated (slower than light speed, c)

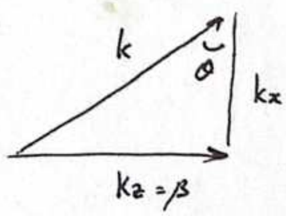
TE → 1 half-cycle → TE_1

TM → 2 half-cycle → TE_2

→ higher have f_c.

$k = 2\pi/\lambda \qquad k_c = 2\pi/\lambda_c. \qquad k_z = \beta = 2\pi/\lambda_g$

change θ

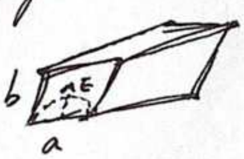


$$\text{ph-v} = \frac{c}{n \cdot \sin\theta}$$

$$\text{g-v} = \frac{c\sin\theta}{n}$$

$$\frac{k_x}{k} = \cos(\theta_m).$$

Rectangular.



TE_{10}: $\quad \lambda_c/2 = a$

TE_{20}: $\quad \lambda_c = a$